JSJ decomposition of knot exteriors, Dehn surgeries, and the L-space conjecture

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Background

Three properties (LO, CTF, NLS)

M is a closed, connected, orientable, irreducible 3-manifold.

(LO) $\pi_1(M)$ is left-orderable:

- $\exists$ a strict total order $<$ on $\pi_1(M)$ satisfying, for any $a, b, c \in \pi_1(M)$,

$$a < b \Leftrightarrow c \cdot a < c \cdot b.$$

- For example
  - $\mathbb{Z}$ is LO.
  - Finite groups are not LO.

Three properties (LO, CTF, NLS)

(CTF) M admits a co-orientable taut foliation.

- A (co-)orientable foliation $\mathcal{F}$ is a decomposition of M into a disjoint union of orientable surfaces.

  Example. A surface bundle over S^1 is foliated by the fibers.

- Every orientable closed 3-manifolds has a orientable foliation (Lickorish, 65).

(NLS) M is not an L-space.

An L-space is a $\mathbb{Q}$-homology sphere who Heegaard Floer homology is "minimal".

Conjecture (Boyer-Gordon-Watson, Juhász, 10s)

M is a connected, compact, orientable, irreducible 3-manifold.

$$LO \iff CTF \iff NLS$$

Known:

- $CTF \implies NLS$

  (Ozsváth-Szabó 04; Bowden, Kazez-Roberts 15)

- Conjecture holds for graph manifolds.

  (Eisenbud-Jankins-Neumann 80s, Naimi 90s; Stipsicz-Lisca 05; Boyer-Clay, Hanselman-Rasmussen-Rasmussen-Watson 15)

L-space conjecture, toroidal manifolds, slope detections

M is a toroidal $\mathbf{Q}$-homology sphere.

- M is called toroidal if $\exists$ an incompressible torus in M.

The basic idea:

- Cut M open along incompressible tori:

$$M = \sqcup M_i.$$

- M has the property $*$ if the $*$-detected slopes are matched along the gluing map, where $* \in \{CTF, LO, NLS\}$. (Hanselman-Rasmussen-Rasmussen-Watson 15, Boyer-Clay 21, Boyer-Gordon-H. 21)

Study LO and NLS surgeries on satellite knots

Cut along a single torus

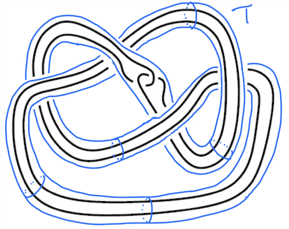


Figure 1: Whitehead double

- Let $K = P(K')$ where K' is the companion knot and P is the pattern.
- We view $K(r) = X(K') \cup_T P(r)$

Theorem (Boyer-Gordon-H. 23)

If P has winding number zero, then $\pi_1(K(r))$ is LO and NLS unless r is a cabling slope of K.

Cut along multiple tori

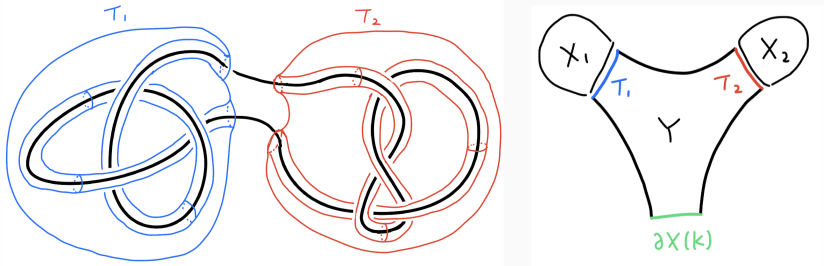


Figure 2: Composite knot

- If $K = K_1 \# K_2$, then

$$K(r) = X_1 \cup_{T_1} Y(r) \cup_{T_2} X_2$$

where $Y(r)$ is the r-filling along $\partial X(K)$.

Cut along multiple tori: the main result

- The JSJ graph of a satellite knot K is the **rooted** tree dual to the JSJ tori. The **root** is the piece that contains $\partial X(K)$.

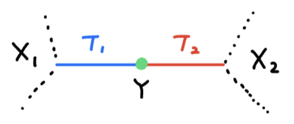


Figure 3: JSJ graph of $K_1 \# K_2$

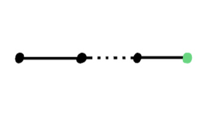


Figure 4: Rooted interval

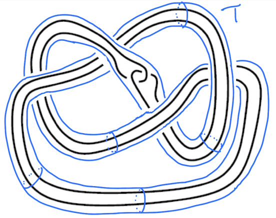
Theorem (Boyer-Gordon-H.)

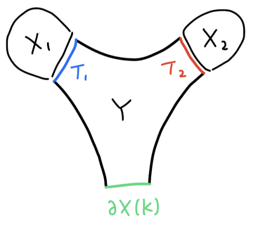
If the JSJ graph of the knot exterior $X(K)$ is not an rooted interval, then $K(r)$ is LO and NLS for any non-cabling slope r.

- All non-meridional surgeries on a composite knot is NLS. (Krcatovich, 15)

CTF surgeries on satellite knots

A key result used in our proof





Theorem (Boyer-Gordon-H. ,21)

Let μ be the knot meridian of a nontrivial knot K in S^3.

- μ is $*$-detected if $* \in \{LO, NLS\}$.
- μ is CTF-detected if K is fibered.

CTF surgeries

- We can prove the same results for CTF surgeries, with an additional fibering assumption.

Conjecture (Boyer-Gordon-H. ,21)

The meridian μ of a nontrivial knot K in S^3 is *CTF*-detected.

- For composite knots,

Theorem (Delman and Roberts, 20)

If $K = K_1 \# K_2$ and $K_i \in \{\text{fibered, alternating, Montesinos}\}$, then $K(r)$ is CTF for any $r \in \mathbb{Q}$.

CTF detection

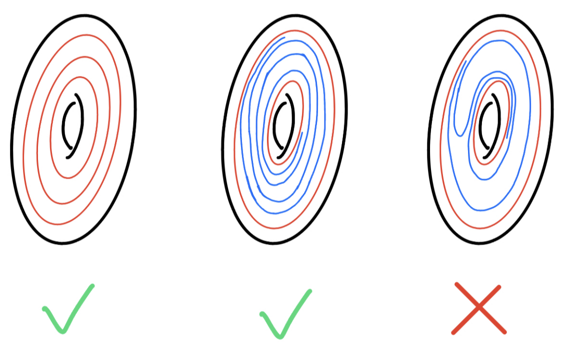


Figure 5: $\mathcal{F} \cap \partial X(K)$

Thank you!