

Slope Detection and Toroidal Manifolds

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Background

Three properties

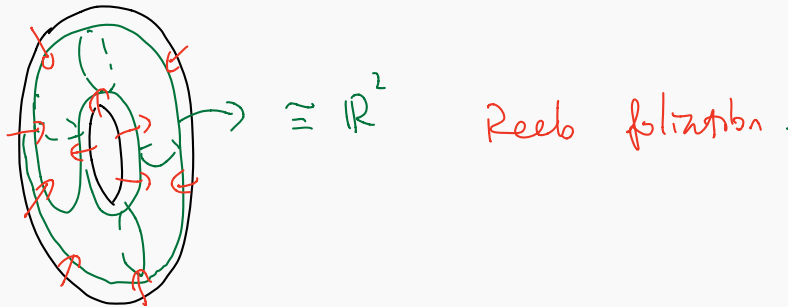
Y is a connected, orientable, irreducible 3-manifold.

(LO) $\pi_1(Y)$ is left-orderable.

\exists a strict total order $<$ on $\pi_1(Y)$ satisfying $a < b \Leftrightarrow c \cdot a < c \cdot b$ for any $a, b, c \in \pi_1(Y)$.

(CTF) Y admits a co-orientable taut foliation.

- A (co-)orientable foliation \mathcal{F} is a decomposition of Y into a disjoint union of orientable surfaces.
- \mathcal{F} is **taut**, if for any $p \in Y$, \exists a transverse loop passing p .



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(NLS) Y is not an L -space.

Y is an L -space, if $|H_1(Y)| < \infty$ and $\widehat{HF}(Y)$ is “minimal”.

The L-space conjecture

Conjecture (Boyer-Gordon-Watson, Juhász)

Y is a connected, compact, orientable, irreducible 3-manifold.

$$LO \iff CTF \iff NLS$$

Known:

- $CTF \implies NLS$ (Ozsváth-Szabó; Kazez-Roberts, Bowden)
- If $b_1(Y) > 0$, then Y is
 - CTF (Gabai);
 - LO (Boyer-Rolfsen-Wiest);
 - NLS by definition.

Example: If $\partial Y \neq \emptyset$ and $Y \neq B^3$, then $b_1(Y) > 0$.

Slope detection

Rough ideas

Y is a toroidal \mathbf{Q} -homology sphere ($b_1(Y) = 0$).

essential $\sqcup T^2 \subset Y$

\downarrow cut Y open along $\sqcup T^2$

$$Y = M_1 \cup M_2 \cup \dots \cup M_n$$

$\partial M_i \neq \emptyset \Rightarrow M_i$ is CTF, LO, NLS

Question:

Under what boundary conditions is Y LO, CTF or NLS?

slope detection (Boyer-Clay, graph manifolds)

Slopes

M is a **knot manifold** (compact, connected, orientable, irreducible, $\partial M \cong T^2$ and $M \neq D^2 \times S^1$.)

A **rational slope** on ∂M is an isotopic class of simple closed curves on $\partial M \cong T^2$.

The **longitudinal slope** λ is unique.

- $[\lambda]$ has order $k > 0$ in $H_1(M)$;
- k copies of λ bounds an orientable surface F in M .

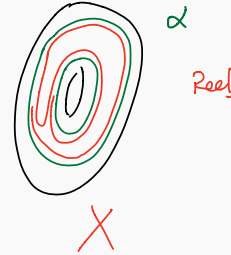
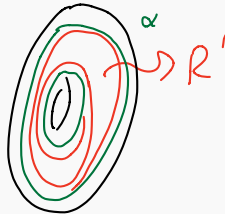
Any slope μ with $|\mu \cdot \lambda| = 1$ is called a **meridional slope**.

Slope detection

M is a knot manifold. A **rational slope** α on ∂M is ***-detected** if the following holds

* = **CTF**:

\exists a CTF \mathcal{F} on M transverse to ∂M so that $\mathcal{F} \cap \partial M$ is **Reebless** and contains a **closed leaf of slope** α .



Slope detection

M is a knot manifold. A **rational slope** α on ∂M is ***-detected** if the following holds

* = **CTF**:

\exists a CTF \mathcal{F} on M transverse to ∂M so that $\mathcal{F} \cap \partial M$ is **Reebless** and contains a **closed leaf of slope** α .

* = **LO**:

\exists a representation $\rho : \pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R})$ satisfying that $\rho|_{\pi_1(\partial M)}$ has **no fixed point**, but $\rho(\alpha)$ has a **fixed point**.

* = **NLS**:

$\alpha \notin$ the interior of **the set of the L-space slopes**.

Basic gluing properties

Theorem

M_1 and M_2 are knot manifolds ($\neq D^2 \times S^1$). $Y = M_1 \cup_f M_2$ is irreducible.
If $f(\alpha_1) = \alpha_2$ with both α_i are $*$ -detected, then Y has property $*$.

- $*$ = NLS;
(Hanselman-Rasmussen-Watson, Rasmussen-Rasmussen)
- $*$ = LO; (Boyer - Clay)
- $*$ = CTF; (Boyer - Gordon - H.)

Conjecture

The converse of the gluing theorem holds.

Known if $*$ = NLS (HRW, RR)

Results

Detected slopes

Theorem (Boyer-Gordon-H.)

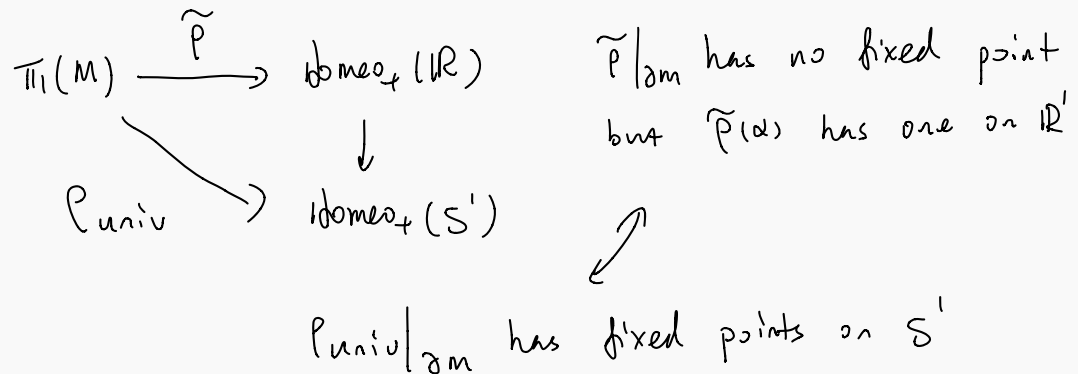
$M \neq D^2 \times S^1$ is an irreducible \mathbb{Z} -homology $D^2 \times S^1$. Then

- all meridional slopes are $*$ -detected if $* \in \{LO, NLS\}$.
- all meridional slopes are CTF-detected if M is **fibred**.

Proof.

When $* = NLS$, we use Rasmussen-Rasmussen.

When $* = LO$,



Detected slopes

Theorem (Boyer-Gordon-H.)

$M \neq D^2 \times S^1$ is an irreducible \mathbb{Z} -homology $D^2 \times S^1$. Then

- all meridional slopes are $*$ -detected if $*$ $\in \{\text{LO}, \text{NLS}\}$.
- all meridional slopes are CTF-detected if M is **fibred**.

Proof.

When $*$ = **NLS**, we use Rasmussen-Rasmussen.

When $*$ = **LO**, we obtain a Universal circle action $\rho : \pi_1(M) \rightarrow \text{Homeo}_+(S^1)$ from a finite depth foliation, satisfying that $\rho|_{\partial M}$ has a **fixed point**.

When $*$ = CTF, one needs a laminar branched surface B in M such that ∂B carries all **meridional slopes**. □

Toroidal \mathbb{Z} -homology sphere

Conjecture (Ozsváth-Szabó)

Y is an irreducible \mathbb{Z} -homology sphere. If Y is an L-space, then Y is either S^3 or the Poincaré sphere.

Theorem (Eftekhary, Hanselman-Rasmussen-Watson)

Y is an irreducible toroidal \mathbb{Z} -homology sphere. Then Y is not an L-space.

Theorem (Boyer-Gordon-H.)

$Y = M_1 \cup_f M_2$ is an irreducible toroidal \mathbb{Z} -homology sphere. Then

- *Y is LO.*
- *Y is CTF if one of M_i is fibered.*

Remark: Similar statements can be made if $|H_1(Y)| \leq 4$.

Cyclic branched covers of toroidal knots

Conjecture (Gordon-Lidman)

The n -fold cyclic branched cover of S^3 over a prime satellite knot is LO, CTF and NLS for $n \geq 2$.

Theorem (Boyer-Gordon-H.)

Suppose that $P(K)$ is a prime satellite knot with pattern P and companion K . Then the n -fold cyclic branched cover of S^3 branched cover $P(K)$ is

- *NLS and LO for $n \geq 2$;*
- *CTF if the companion knot K is fibered for $n \geq 2$.*

An application to knot theory

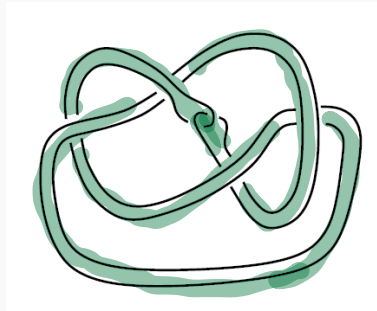


Figure 1: A satellite knot

Theorem (Menasco)

A prime satellite link is not alternating.

Theorem (Boyer-Gordon-H.)

A prime satellite link is not quasi-alternating.

Thank you